

# ON GEOMETRIC ANALYSIS OF THE DYNAMICS OF VOLUMETRIC EXPANSION

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**ABSTRACT.** In this paper, we discuss the global aspect of the geometric dynamics of volumetric expansion and its application to the problem of the existence in the space-time of compact and complete spacelike hypersurface.

## 1. INTRODUCTION

In this paper, we discuss the global aspect of the dynamics of volumetric expansion and its connection with the problem on the existence of spacelike closed and complete hypersurfaces in the space-time. This problem is interesting for several reasons. First, manifolds that admit global spacelike hypersurfaces are more relevant to physics than manifolds in which such hypersurfaces cannot exist (see [1, p. 207]). Second, the problem on the existence of closed spacelike hypersurfaces (see [2], [3, p. 164]) is closely related to the accelerated expansion of the Universe (see [4]).

In Sec. 2, we recall basic definitions and elementary facts of the dynamics of volumetric expansion in the case of the absence of a metric; here we use the methods proposed in [5]. In Sec. 3, we examine this problem for complete Riemannian manifolds by the methods developed in [6, 7]. Finally, in Secs. 4 and 5 we discuss applications of the dynamics of volumetric expansion to the problem on the existence of compact, complete, spacelike hypersurfaces in Lorentzian manifolds and in the space-time.

## 2. DYNAMICS OF VOLUMETRIC EXPANSION WITHOUT METRIC

Let  $M$  be a differentiable manifold of dimension  $n$  ( $n \geq 2$ ) and  $L(M)$  be the bundle of linear frames over  $M$  with the structure group  $\mathrm{GL}(n, \mathbb{R})$ . We define an  $\mathrm{SL}(n, \mathbb{R})$ -structure on the manifold  $M$  as an  $\mathrm{SL}(n, \mathbb{R})$ -subbundle of the bundle  $L(M)$  for the subgroup  $\mathrm{SL}(n, \mathbb{R})$  of  $\mathrm{GL}(n, \mathbb{R})$ . It is well known (see [8, p. 13]) that the  $\mathrm{SL}(n, \mathbb{R})$ -structure on  $M$  is nothing but the volume element  $\omega$ , i.e., an  $n$ -form  $\omega$ , which does not vanish everywhere on  $M$ . Moreover, it is known that a connected manifold  $M$  admits an  $\mathrm{SL}(n, \mathbb{R})$ -structure if and only if it is orientable. Hence, we assume in the sequel that the  $n$ -form  $\omega$  satisfies the condition

$$\omega\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) > 0$$

for an arbitrary local coordinate system  $x^1, \dots, x^n$  matched with the orientation of  $M$  (see [9, p. 259], [10, p. 86]).

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The volume element  $\omega$  on the manifold  $M$  allows one to introduce the integral  $\int_M f := \int_M f \omega$  of an arbitrary compactly supported in  $M$ , differentiable function  $f: M \rightarrow \mathbb{R}$  (see [10, p. 87]). In particular, if the manifold  $M$  is closed (i.e., compact without boundary), then we can define its *volume* as follows (see [10, p. 87]):

$$\text{Vol}_\omega(M) := \int_M 1 = \int_M \omega > 0.$$

Let  $\xi$  be a differentiable vector field on  $M$ . It is well known (see [9, pp. 22-23], [11, p. 140], [12, pp. 27, 29]) that in a neighborhood  $U$  of each point of the manifold  $M$  the field  $\xi$  generates a *local flow*, which is a local one-parameter group of infinitesimal diffeomorphisms or, in other words, transformations  $\phi_t(x): U \rightarrow M$ . Such transformations are given by the formula

$$\phi_t(x^k) = \bar{x}^k = x^k + t\xi^k$$

in an arbitrary local coordinate system  $x^1, \dots, x^n$  in a neighborhood  $U$ , where  $t \in (-\varepsilon, +\varepsilon) \subset \mathbb{R}$  is a parameter and  $\xi = \xi^k \partial_k$ . The converse assertions is also valid (see [9, pp. 21-22], [11, p. 140]), namely, a local flow (or, in other words, a local one-parameter group of infinitesimal transformations of the manifold  $M$  consisting of diffeomorphisms  $\phi_t(x): U \rightarrow M$ ), for some open set  $U \subset M$ , an interval  $(-\varepsilon, +\varepsilon) \subset \mathbb{R}$ , and arbitrary  $t \in (-\varepsilon, +\varepsilon)$  and  $x \in U$ , induces a vector field  $\xi$  on  $U$  as follows. At each point  $x \in U$ , we define a vector  $\xi_x$  tangent to the curve  $x(t) = \phi_t(x)$  and such that  $\xi^k = dx^k/dt$  for  $k = 1, \dots, n$  in a local coordinate system  $x^1, \dots, x^n$  in  $U$ . The curve  $x(t) = \phi_t(x)$  is called the *trajectory* of the flow. If the transformation group is *global*, then the vector field  $\xi$  generated by it is said to be *complete* (see [9, p. 22], [12, p. 29]). In particular, on a compact manifold  $M$  each differentiable vector field  $\xi$  is complete (see [9, p. 23]).

The vector field  $\xi$  is also called the *velocity vector* (field) of the flow. For an arbitrary differentiable tensor field  $T$ , one can consider its *Lie derivative* along trajectories of the flow with the velocity vector  $\xi$ :

$$L_\xi T := \frac{d}{dt} (\phi_t^* T)|_{t=0}$$

(see [9, p. 36], [10, p. 71]). It is well known (see [13, p. 211]) that the Lie derivative  $L_\xi T$  measures the *rate of the change of the tensor*  $T$  under deformations determined by the one-parameter group of differentiable transformations  $\phi_t$  generated by the vector field  $\xi$ . In particular, the rate  $L_\xi \omega$  of the change of the volume element  $\omega$  or, in other words, the *rate of volumetric expansion* under deformations determined by the one-parameter group of differentiable transformations  $\phi_t$  generated by the field  $\xi$  can be calculated by the formula (see [9, p. 259], [13, p. 212])

$$(1) \quad L_\xi \omega := (\text{div}_\omega \xi) \cdot \omega.$$

Due to (1), the function  $\text{div}_\omega \xi$  is called the *logarithmic rate of change of the volume* (or the *rate of volumetric expansion*) along the flow generated by the field  $\xi$  (see [12, p. 195]). On the other hand, for a vector field  $\xi$  with compact support in  $M$ , *Green's theorem* is valid (see [9, p. 259]):

$$(2) \quad \int_M (\text{div}_\omega \xi) \cdot \omega = 0.$$

Obviously, the conditions  $\text{div}_\omega \xi > 0$  and  $\text{div}_\omega \xi < 0$  contradict (2). If  $\text{div}_\omega \xi \geq 0$  or  $\text{div}_\omega \xi \leq 0$ , then Eq. (2) implies that  $\text{div}_\omega \xi = 0$ . This means that  $L_\xi \omega = 0$ ,

i.e., the one-parameter group of differentiable transformations  $\phi_t$  leaves  $\omega$  invariant and the vector field  $\xi$  is an *infinitesimal automorphism* of the  $\text{SL}(n, \mathbb{R})$ -structure (see [8, pp. 9-10]). In hydrodynamics (even in the absence of a metric), such a vector field  $\xi$  is said to be *divergence-free* and the flow generated by it is said to be *incompressible* (see [14, p. 125]). The following assertion is obvious.

**Proposition 1.** *Let  $(M, \omega)$  be a connected differentiable manifold equipped with an  $\text{SL}(n, \mathbb{R})$ -structure and let a flow with a compactly supported velocity vector  $\xi$  be given on  $M$ . The volume element  $\omega$  cannot increase (decrease) along trajectories of the flow. If the volume element  $\omega$  is a nondecreasing (or nonincreasing) function along trajectories of the flow, then this flow is incompressible and its rate of volumetric expansion is equal to zero.*

*Proof.* At each point of the manifold  $M$ , the signs of the functions of the logarithmic rate  $\text{div}_\omega \xi$  and the rate  $L_\xi \omega$  of volumetric expansion coincide. Moreover,  $\text{div}_\omega \xi = 0$  if and only if  $L_\xi \omega = 0$ . This means that if the velocity vector  $\xi$  has a compact support in the manifold  $M$  with the volume element  $\omega$ , then the assertion of the theorem is a direct consequence of Green's theorem.  $\square$

Since the Lie derivative  $L_\xi \omega$  of the volume element  $\omega$  measures the rate of its change under the action of the group of differentiable transformations  $\phi_t$  generated by the field  $\xi$ , the Lie derivative  $L_\xi(L_\xi \omega)$ , in its turn, measures the *acceleration* of volumetric expansion, i.e., the *acceleration* of the change of the volume element  $\omega$  along trajectories of the flow with the velocity vector  $\xi$ . In this case, the following relation holds:

$$(3) \quad \begin{aligned} L_\xi(L_\xi \omega) &= L_\xi((\text{div}_\omega \xi) \cdot \omega) = (L_\xi(\text{div}_\omega \xi)) \cdot \omega + (\text{div}_\omega \xi) \cdot L_\xi \omega \\ &= (L_\xi(\text{div}_\omega \xi) + (\text{div}_\omega \xi)^2) \cdot \omega, \end{aligned}$$

where the function  $L_\xi(\text{div}_\omega \xi)$  characterizes the rate of the change of the logarithmic rate of volumetric expansion  $\text{div}_\omega \xi$  along trajectories of the flow generated by the field  $\xi$ . Obviously, the vanishing of the acceleration of volumetric expansion  $L_\xi(L_\xi \omega)$  leads to the condition  $L_\xi(\text{div}_\omega \xi) = -(\text{div}_\omega \xi)^2 \leq 0$ . As a result, the logarithmic rate of volumetric expansion either decreases or vanishes along trajectories of the flow. On the other hand, if the logarithmic rate of volumetric expansion is a nondecreasing (or even zero) function along trajectories of the flow, then the rate of volumetric expansion of this flow is also a nondecreasing function. Obviously, the growth condition  $L_\xi(\text{div}_\omega \xi) > 0$  for the logarithmic rate implies the increasing of the rate of volumetric expansion since in this case  $L_\xi(L_\xi \omega) > 0$ .

The vector field  $(\text{div}_\omega \xi)\xi$  is called the *vector of logarithmic rate of volumetric expansion*. Based on Eq. (3), we can prove the following assertion.

**Proposition 2.** *Let  $(M, \omega)$  be a connected differentiable manifold equipped with an  $\text{SL}(n, \mathbb{R})$ -structure and let a flow with velocity vector  $\xi$  be given on  $M$  such that the vector of logarithmic rate of volumetric expansion  $(\text{div}_\omega \xi)\xi$  has a compact support in  $M$ . The rate of volumetric expansion of this flow cannot increase (decrease) along trajectories. If the rate  $L_\xi \omega$  of volumetric expansion along trajectories of the flow is a nondecreasing (or nonincreasing) function, then this flow has a constant rate of volumetric expansion. In particular, if the logarithmic rate of volumetric expansion  $\text{div}_\omega \xi$  is a nondecreasing function along trajectories of the flow, then the flow is incompressible and its rate of volumetric expansion  $L_\xi \omega$  is equal to zero.*

*Proof.* Let  $M$  be a differentiable manifold with volume element  $\omega$ . First, we note that for the vector field  $(\operatorname{div}_\omega \xi)\xi$ , which is compactly supported in  $M$ , Green's theorem has the form

$$(4) \quad \int_M \operatorname{div} ((\operatorname{div}_\omega \xi)\xi) \cdot \omega = \int_M \left( L_\xi(\operatorname{div}_\omega \xi) + (\operatorname{div}_\omega \xi)^2 \right) \cdot \omega = 0.$$

Second, it is easy to see from (3) that the inequality

$$L_\xi(L_\xi \omega) > 0 \quad (L_\xi(L_\xi \omega) < 0),$$

which is valid everywhere on  $M$ , implies the inequality

$$L_\xi(\operatorname{div}_\omega \xi) + (\operatorname{div}_\omega \xi)^2 > 0 \quad (L_\xi(\operatorname{div}_\omega \xi) + (\operatorname{div}_\omega \xi)^2 < 0),$$

which contradicts (4). Similarly, from (3) we also conclude that the inequality

$$L_\xi(L_\xi \omega) \geq 0 \quad (L_\xi(L_\xi \omega) \leq 0),$$

which is valid everywhere on  $M$ , implies the inequality

$$L_\xi(\operatorname{div}_\omega \xi) + (\operatorname{div}_\omega \xi)^2 \geq 0 \quad (L_\xi(\operatorname{div}_\omega \xi) + (\operatorname{div}_\omega \xi)^2 \leq 0).$$

Therefore, Eq. (4) implies that

$$L_\xi(\operatorname{div}_\omega \xi) + (\operatorname{div}_\omega \xi)^2 = 0$$

and hence

$$L_\xi(L_\xi \omega) = 0.$$

In particular, from (4) for  $L_\xi(\operatorname{div}_\omega \xi) \geq 0$  we conclude that  $\operatorname{div}_\omega \xi = 0$  and hence the equality  $L_\xi \omega = 0$  holds.  $\square$

The following assertion is valid.

**Proposition 3.** *On a closed (i.e., compact without boundary) differentiable manifold  $(M, \omega)$  equipped with an  $\operatorname{SL}(n, \mathbb{R})$ -structure, there are no flows with velocity vector  $\xi$  and the nondecreasing along the flow logarithmic rate of volumetric expansion  $\operatorname{div}_\omega \xi$ , if the acceleration  $L_\xi(\operatorname{div}_\omega \xi) > 0$  at least at one point.*

This assertion is valid since the conditions imposed on the scalar functions  $\operatorname{div}_\omega \xi$  and  $L_\xi(\operatorname{div}_\omega \xi)$  contradict Eq. (4), which is also valid on any closed (i.e., compact without boundary) manifold  $M$ .

### 3. DYNAMICS OF VOLUMETRIC EXPANSION ON COMPLETE RIEMANNIAN MANIFOLDS

Let  $\mathcal{O}(n, \mathbb{R})$  be a subgroup of orthogonal transformations from  $\operatorname{GL}(n, \mathbb{R})$ . We determine an  $\mathcal{O}(n, \mathbb{R})$ -structure on an  $n$ -dimensional ( $n \geq 2$ ) differentiable manifold  $M$  as an  $\mathcal{O}(n, \mathbb{R})$ -subbundle of the bundle  $L(M)$ . It is well known (see [8, p. 13]) that an  $\mathcal{O}(n, \mathbb{R})$ -structure on  $M$  is nothing but the Riemannian metric  $g$  on  $M$ . In this case the pair  $(M, g)$  is called a Riemannian manifold. For a Riemannian manifold  $(M, g)$ , we can introduce the canonical volume element  $\omega = dv$ , which is defined in an arbitrary oriented local coordinate system  $x^1, \dots, x^n$  on  $M$  by the formula  $dv = \sqrt{\det g} dx^1 \wedge \dots \wedge dx^n$ .

Recall that a Riemannian manifold  $(M, g)$  is complete if each geodesic in  $(M, g)$  can be extended for arbitrarily large values of its canonical parameter (see [9, p. 166]). It is known that on each connected differentiable manifold there exists a structure of a complete Riemannian manifold (see [15, p. 175]).

On a complete Riemannian manifold, the *generalized Green's theorem* holds (see [16, 17]). Namely, for a differentiable vector field  $X$  on a complete (noncompact) oriented Riemannian manifold  $M$ , the conditions  $\operatorname{div} X \geq 0$  (or  $\operatorname{div} X \leq 0$ ) and  $|X| \in L^1(M, g)$  imply  $\operatorname{div} X = 0$ ; here  $|X| = \sqrt{g(X, X)}$  and  $\operatorname{div} X = \operatorname{div}_\Omega X$  for  $\Omega = dv$ . Taking into account this theorem, we can reformulate Proposition 1 as follows.

**Corollary 1.** *Let on a complete noncompact oriented Riemannian manifold  $(M, g)$ , a flow with complete field of velocity vectors  $\xi$  be given such that  $|\xi| \in L^1(M, g)$ . If the volume element of the manifold is a nondecreasing (or nonincreasing) function along trajectories of the flow, then the flow is incompressible and its rate of volumetric expansion is equal to zero.*

If we take the vector of logarithmic rate  $(\operatorname{div} \xi)\xi$  as  $X$  in the generalized Green's theorem, then Proposition 2 implies the following assertion.

**Corollary 2.** *Let on a complete, noncompact, oriented Riemannian manifold  $(M, g)$ , a flow with complete field of velocity vectors  $\xi$  be given such that  $|(\operatorname{div} \xi)\xi| \in L^1(M, g)$ . If the rate of volumetric expansion is a nondecreasing (or nonincreasing) function along trajectories of the flow, then this flow has a constant rate of volumetric expansion. In particular, if the logarithmic rate of volumetric expansion is a nondecreasing function along trajectories of the flow, then the flow is incompressible and its rate of volumetric expansion is equal to zero.*

As an example, we consider a local one-parameter group of infinitesimal transformations (i.e., a flow) generated by the vector field  $\xi$  of a Ricci soliton, i.e., a Riemannian manifold  $(M, g)$  such that

$$-2 \operatorname{Ric} = L_\xi g + 2\lambda g,$$

where  $\operatorname{Ric}$  is the Ricci tensor of the metric  $g$ ,  $L_\xi g$  is the derivative of the metric  $g$  in the direction of the field  $\xi$ , and  $\lambda$  is a constant (see [18, p. 22]). For the vector of logarithmic rate of volumetric expansion  $(\operatorname{div} \xi)\xi$ , we have (see [19])

$$(5) \quad \operatorname{div}(\operatorname{div} \xi)\xi = -L_\xi s + (s + n\lambda)^2,$$

where  $s = \operatorname{trace}_g \operatorname{Ric}$  is the scalar curvature of the metric  $g$ .

**Proposition 4.** *Let  $(M, g)$  be a complete, noncompact, oriented Ricci soliton with a complete vector field  $\xi$  such that  $|(\operatorname{div} \xi)\xi| \in L^1(M, g)$ . If the scalar curvature  $s$  of the Ricci soliton is a nonincreasing function along trajectories of the flow generated by the field  $\xi$ , then  $s = -n\lambda$  and the flow is incompressible.*

#### 4. DYNAMICS OF VOLUMETRIC EXPANSION ON LORENTZIAN MANIFOLDS

On an  $n$ -dimensional ( $n \geq 3$ ) differentiable manifold  $M$ , we introduce an  $\mathcal{O}(1, n-1)$ -structure as an  $\mathcal{O}(1, n-1)$ -subbundle of the bundle  $L(M)$ , where  $\mathcal{O}(1, n-1)$  is the subgroup of Lorentz transformations in  $\operatorname{GL}(n, \mathbb{R})$ . It is well known (see [8, p. 13]) that an  $\mathcal{O}(1, n-1)$ -structure on  $M$  is nothing but a pseudo-Riemannian metric  $g$  on  $M$  with Lorentz signature  $(-++ \dots +)$ . It is also known that any noncompact differentiable manifold admits an  $\mathcal{O}(1, n-1)$ -structure, i.e., possesses a metric with Lorentz signature. An  $\mathcal{O}(1, n-1)$ -structure on a compact manifold exists if and only if its Euler characteristics is equal to zero (see [20, p. 50]). A pair  $(M, g)$  is called a *Lorentzian manifold* (see [20, p. 50]). The canonical volume element

on  $(M, g)$  has the form  $dv = \sqrt{|\det g|} dx^1 \wedge \cdots \wedge dx^n$  for an arbitrary oriented local coordinate system  $x^1, \dots, x^n$  on  $M$ .

In on an  $n$ -dimensional ( $n \geq 3$ ) Lorentzian manifold  $(M, g)$  a complete unit timelike vector field  $\xi$  is given, then  $(M, g)$  is said to be *time oriented by the field  $\xi$*  (see [20, p. 50]). In this case, the following consequence of Proposition 2 is valid.

**Corollary 3.** *Assume that on an  $n$ -dimensional ( $n \geq 3$ ) Lorentzian manifold  $(M, g)$  there exists a flow with a unit timelike compactly supported velocity vector  $\xi$ . The rate of volumetric expansion of this flow cannot be a function that increases (decreases) along its trajectories and, in particular, the logarithmic rate of volumetric expansion cannot increase.*

Assume that in  $(M, g)$  there exists a spacelike hypersurface  $M'$  (see [20, p. 92]) orthogonal to trajectories of the flow generated by the vector field  $\xi$ . In this case, the *acceleration vector* of the flow  $\dot{\xi} = \nabla_\xi \xi$  is a tangent vector for  $M'$ . At points of  $M'$ , the following relations holds (see [4, 21]):

$$(6) \quad \operatorname{div} \dot{\xi} = \operatorname{Ric}(\xi, \xi) + g(\sigma, \sigma) + (n-1)^{-1}(\operatorname{div} \xi)^2 + L_\xi(\operatorname{div} \xi),$$

where  $g(\sigma, \sigma) \geq 0$  is the square of the traceless part of the *second fundamental form* of the hypersurface  $M'$  (see [20, p. 93]). The following theorem is valid.

**Theorem 1.** *Assume that in an  $n$ -dimensional ( $n \geq 3$ ) Lorentzian manifold  $(M, g)$ , there exist a spacelike hypersurface  $M'$  and a flow orthogonal to it with unit timelike velocity vector  $\xi$ . If  $\operatorname{Ric}(\xi, \xi) \geq 0$  and for the logarithmic rate of volumetric expansion  $\operatorname{div} \xi$  we have  $L_\xi(\operatorname{div} \xi) \geq 0$  at each point of  $M'$  and, moreover, at least at one point  $L_\xi(\operatorname{div} \xi) > 0$ , then the hypersurface  $M'$  cannot be closed.*

*Proof.* Assume that the hypersurface  $M'$  is a closed (i.e., compact without boundary) spacelike submanifold in  $(M, g)$ , which is orthogonal at each point to the unit timelike vector field  $\xi$ . Applying Green's theorem to the vector field  $\dot{\xi} = \nabla_\xi \xi$ , which is tangent to  $M'$ , we obtain the integral relation

$$(7) \quad \int_{M'} \left( \operatorname{Ric}(\xi, \xi) + g(\sigma, \sigma) + (n-1)^{-1}(\operatorname{div} \xi)^2 + L_\xi(\operatorname{div} \xi) \right) \cdot dv' = 0,$$

where  $dv'$  is the volume element of  $M'$ . Obviously, the conditions  $\operatorname{Ric}(\xi, \xi) \geq 0$  and  $L_\xi(\operatorname{div} \xi) \geq 0$  everywhere on  $M'$  and  $L_\xi(\operatorname{div} \xi) > 0$  at least at one point of  $M'$  contradict the integral relation (7).  $\square$

*Remark 1.* We recall the following classical result (see [2]): in an  $n$ -dimensional ( $n \geq 3$ ) compact space-time with the Ricci tensor satisfying the condition  $\operatorname{Ric}(X, X) > 0$  for all nonspacetime vectors  $X \neq 0$ , there are no closed (i.e., compact without boundary) spacelike hypersurfaces.

If we assume that the spacelike hypersurface  $M'$  is a complete (noncompact) oriented Riemannian manifold, then using the generalized Green's theorem we can prove the following.

**Theorem 2.** *Assume that in an  $n$ -dimensional ( $n \geq 3$ ) Lorentzian manifold  $(M, g)$  there exist a spacelike complete (noncompact) oriented hypersurface  $M'$  and a flow orthogonal to it with unit timelike velocity vector  $\xi$  such that  $|\dot{\xi}| \in L^1(M, g)$ . If  $\operatorname{Ric}(\xi, \xi) \geq 0$  and the logarithmic rate of volumetric expansion  $\operatorname{div} \xi$  satisfies the inequality  $L_\xi(\operatorname{div} \xi) \geq 0$  at each point of  $M'$ , then  $M'$  is a completely geodesic submanifold of  $(M, g)$ .*

*Proof.* By the condition of the theorem, the vector field  $\dot{\xi} = \nabla_{\xi}\xi$  is tangent for the complete, noncompact, oriented Riemannian manifold  $M'$  on which Eq. (5) holds. If we impose the conditions  $\text{Ric}(\xi, \xi) \geq 0$  and  $L_{\xi}(\text{div } \xi) \geq 0$  everywhere on  $M'$ , then due to (5) the inequality  $\text{div } \dot{\xi} \geq 0$  also holds. In this case, by the generalized Green's theorem we conclude that  $\text{div } \dot{\xi} = 0$ . Then, in particular, from (5) we see that  $\text{div } \xi = \sigma = 0$ . This means (see [4, 21]) that the second fundamental form of the hypersurface  $M'$  vanishes and hence the hypersurface  $M'$  itself is a *completely geodesic submanifold* of  $(M, g)$  (see [20, pp. 93-94]).  $\square$

## 5. DYNAMICS OF VOLUMETRIC EXPANSION IN SPACE-TIME

Recall that a *space-time* is a connected four-dimensional oriented Lorentzian manifold  $(M, g)$  (see [22, c. 27]). For the case  $n = 4$ , Eq. (5) follows from the Landau–Raychaudhuri equation (see [22, pp. 97–98]), which describes the dynamics of flows of the cosmological liquid in the space-time. Streamlines of this liquid are trajectories of the flow generated by a unit timelike vector field  $\xi$  (see [20, p. 47] and [23, p. 92]). The hydrodynamical sense of variables appearing in Eq. (5) is as follows (see [1, p. 219] and [22, p. 96]):  $\sigma$  is the tensor of transversal shear,  $\theta$  is the volumetric divergence, and  $\dot{\xi} = \nabla_{\xi}\xi$  is the vector field of the 4-acceleration of the cosmological liquid. In the case of the *perfect liquid* we have  $\text{Ric}(\xi, \xi) = 4\pi(\mu + 3\rho)$ , where  $\mu$  is the energy density and  $\rho$  is the pressure (see [22, p. 98]). Moreover, in the case  $n = 4$ , following Hawking and Penrose (see [24, p. 539]), the inequality  $\text{Ric}(X, X) \geq 0$ , which is valid for all unit timelike vectors  $X$ , is called the *energy condition* for the space-time.

We also recall that the increasing of the logarithmic rate of volumetric expansion implies the increasing of the rate of volumetric expansion; this is directly related to the problem on the accelerated expansion of the Universe. In the case of a four-dimensional space-time, the last two theorems acquire a physical content. The following two assertions are valid (cf. [2] and [3, p. 164]).

**Corollary 4.** *Assume that in a space-time satisfying the energy condition, there exist a spacelike hypersurface and a flow of the cosmological liquid orthogonal to it. If the logarithmic rate of volumetric expansion is a nondecreasing function along streamlines and on the hypersurface there exists growth points, then the hypersurface cannot be closed.*

**Corollary 5.** *Assume that in a four-dimensional space-time  $(M, g)$  satisfying the energy condition, there exists a spacelike complete oriented hypersurface and a flow of the cosmological liquid orthogonal to it with velocity vector  $\xi$  such that  $|\dot{\xi}| \in L^1(M, g)$ . If the logarithmic rate of volumetric expansion along streamlines is a nondecreasing (or even zero) function, then the hypersurface is completely geodesic.*

Each closed, oriented three-dimensional manifold  $M'$  is the boundary of a certain four-dimensional manifold (see [25, 26]). In our case, this means that for a closed, oriented, spacelike hypersurface  $M'$ , there exists a four-dimensional submanifold  $N \subset M$  such that  $\partial N = M'$ . Since the space-time  $(M, g)$  is oriented, the submanifold  $N$  is also oriented. In this case, we can assume that the hypersurface  $M'$  is also oriented and its orientation is induced by an imaginary unit normal vector  $\mathcal{N}$  directed outward at each point  $x \in M'$ . In this case, the divergence theorem

(see [27]) has the form

$$\int_N (\operatorname{div} \xi) dv = - \int_{M'} g(\xi, \mathcal{N}) dv' = \int_{M'} dv' = \operatorname{Vol}(M') > 0,$$

where  $\xi = \mathcal{N}$  at each point  $x \in M'$ . In its turn, the divergence theorem for the vector field  $(\operatorname{div} \xi)\xi$  has the form

$$(8) \quad \int_N \left( L_\xi(\operatorname{div} \xi) + (\operatorname{div} \xi)^2 \right) dv = \int_{M'} (\operatorname{div} \xi) dv'.$$

Now it is easy to see that the conditions  $L_\xi(\operatorname{div} \xi) \geq 0$  at all points of  $N$  and  $\operatorname{div} \xi = 0$  at all points of  $M'$  contradict Eq. (8). The following theorem is valid.

**Theorem 3.** *Assume that in the space-time there exist a closed spacelike hypersurface and an outward flow of the cosmological liquid orthogonal to it. This flow cannot simultaneously satisfy the following two conditions:*

- (1) *along streamlines on the submanifold whose boundary coincides with this hypersurface, the acceleration of volumetric expansion (or the logarithmic rate of volumetric expansion) is a nondecreasing function and there exist points at which it is nonzero;*
- (2) *on the hypersurface, the logarithmic rate of volumetric expansion vanishes.*

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#### REFERENCES

- [1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, Vol. 2, W. H. Freeman and Co., San Francisco (1973).
- [2] G. J. Galloway, “Some global aspects of compact space-times,” *Arch. Math.*, **42**, No. 2, 168–172 (1984).
- [3] R. Penrose, *Structure of Space-Time*, W. A. Benjamin, New York–Amsterdam (1968).
- [4] S. E. Stepanov and J. Mikeš, “The generalized Landau–Raychaudhuri equation and its applications,” *Int. J. Geom. Meth. Mod. Phys.*, **12**, No. 8, 1560026 (2015).
- [5] S. E. Stepanov, “Bochner’s technique for an  $m$ -dimensional compact manifold with an  $\operatorname{SL}(m, \mathbb{R})$ -structure,” *Algebra Anal.*, **10**, No. 4, 192–209 (1998).
- [6] I. A. Alexandrova, J. Mikeš, S. E. Stepanov, and I. I. Tsyganok, “Liouville-type theorems in the theory of mappings of complete Riemannian manifolds,” *Vestn. Novosib. Univ. Ser. Mat. Mekh. Inform.*, **15**, No. 3, 3–10 (2015).
- [7] S. E. Stepanov and I. I. Tsyganok, “Harmonic transforms of complete Riemannian manifolds,” *Mat. Zametki*, **100**, No. 3, 441–449 (2016).
- [8] Sh. Kobayashi, *Transformation Groups in Differential Geometry*, Springer-Verlag, Berlin–Heidelberg–New York (1972).
- [9] Sh. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. 1, Interscience, New York–London (1963).
- [10] L. Godinho and J. Natario, *An Introduction to Riemannian Geometry with Applications to Mechanics and Relativity*, Springer-Verlag, Heidelberg–New York–London (2014).
- [11] L. W. Tu, *An Introduction to Manifolds*, Springer Science+Business Media, New York (2008).
- [12] B. O’Neil, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, San Diego (1983).
- [13] B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, *Modern Feometry—Methods and Applications*, Part 1, Grad. Texts Math., **93**, Springer-Verlag, New York etc. (1992).



- [14] V. I. Arnold and B. A. Khesin, *Topological Methods in Hydrodynamics*, Springer-Verlag, New York (1998).
- [15] D. Gromoll, W. Klingenberg, and W. Meyer, *Riemannsche Geometrie im Großen*, Springer-Verlag, Berlin–Heidelberg–New York (1968).
- [16] A. Caminha, P. Souza, and F. Camargo, “Complete foliations of space forms by hypersurfaces,” *Bull. Braz. Math. Soc. New Ser.*, **41**, No. 3, 339–353 (2010).
- [17] A. Caminha, “The geometry of closed conformal vector fields on Riemannian spaces,” *Bull. Braz. Math. Soc. New Ser.*, **42**, No. 2, 277–300 (2011).
- [18] B. Chow and D. Knopf, *The Ricci Flow: An Introduction*, Am. Math. Soc., New York (2004).
- [19] S. E. Stepanov and V. N. Shelepova, “A note on Ricci solitons,” *Math. Notes*, **86**, No. 3, 447–450 (2009).
- [20] J. K. Beem, P. E. Ehrlich, and K. L. Easley, *Global Lorentzian Geometry*, Marcel Dekker, New York (1996).
- [21] S. E. Stepanov, “An analytic method in general relativity,” *Theor. Math. Phys.*, **122**, No. 3, 402–414 (2000).
- [22] R. K. Sachs and H. Wu, *General Relativity for Mathematicians*, Springer-Verlag, New York (1977).
- [23] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge Univ. Press, Cambridge (1973).
- [24] S. W. Hawking and R. Penrose, “The singularities of gravitational collapse and cosmology,” *Proc. Roy. Soc. A*, **314**, 529–548 (1970).
- [25] L. Guillou and A. Marin, *A la Recherche de la Topologie Perdue*, Birkhäuser, Boston–Basel–Stuttgart (1986).
- [26] F. Costantino and D. Thurston, “3-Manifolds efficiently bound 4-manifolds,” *J. Topol.*, **1**, No. 3, 703–745 (2008).
- [27] B. Unal, “Divergence theorems in semi-Riemannian geometry,” *Acta Appl. Math.*, **40**, 173–178 (1995).

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